Indian Statistical Institute, Bangalore
B.Math (Hons.) I Year, Second Semester
Mid-Sem Examination
Real Analysis II

Time: 150 minutes | February 25, 2011 | Instructor: C.R.E.Raja

Maximum marks: 40.

Answer any five and each question is worth 8 Marks.

1. For $v \in \mathbb{R}^k$ and $1 \leq i \leq k$, let $v(i)$ denote the i-th coordinate of v.

   (a) Prove that a sequence (v_n) in $\mathbb{R}^k$ converges iff $(v_n(i))$ converges for every i.

   (b) Prove that every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

2. (a) Let E be a subset of a metric space X. Prove that E' is closed.

   (b) If $E_1, \cdots, E_k$ are closed in $\mathbb{R}$, then prove that $E_1 \times \cdots \times E_k$ is closed in $\mathbb{R}^k$.

3. (a) Let (x_n) be a Cauchy sequence in a metric space X. If (x_n) has a convergent subsequence, then prove that (x_n) converges.

   (b) Prove that a compact metric space X is complete.

   (c) Let E be a non-empty subset of a complete metric space (X, d). Then prove that (E, d) is complete iff E is closed in X.

4. (a) Prove that a subset E of $\mathbb{R}$ is connected if and only if E is an interval.

   (b) Prove that a non-empty countable connected subset of $\mathbb{R}^k$ is singleton.

5. Let $E \subset \mathbb{R}^k$, $E \neq \emptyset$ and define $f(v) = \inf_{x\in E} d(x, v) = \inf_{x\in E} ||x-v||$ for $v \in \mathbb{R}^k$.

   (a) Prove that E is closed iff $\{v \in \mathbb{R}^k \mid f(v) = 0\} = E$.

   (b) If E is compact, then prove that for each $v \in \mathbb{R}^k$ there exists $w_v \in E$ so that $f(v) = d(v, w_v) = ||v - w_v||$.

6. Let $f: X \to Y$ be a continuous function and $E \subset X$.

   (a) If E is compact, then prove that $f(E)$ is compact.

   (b) Suppose X is compact. Then prove that E is compact iff E is closed in X.

   (c) If X is compact and E is closed, then prove that $f(E)$ is closed in Y.

7. (a) If $f: X \to Y$ is continuous and $E \subset X$, then prove that $f(\overline{E}) \subset \overline{f(E)}$.

   (b) Let (K_n) be a decreasing sequence of non-empty compact sets in a metric space X. If $\text{diam}(K_n) \not\to 0$, then prove that $\cap K_n$ has at least two elements.